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Some results on mathematical seriation with applications

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5.1 Introduction

Two examples of abundance matrices in seriated form are shown in Tables 5.1 and 5.2. The first is of five proportions of painted pottery contained in a stratigraphic sequence of levels, one below the other, from an undisturbed refuse mound at Awatovi, Arizona (Burgh 1959). The second is of seven proportions of different types of microliths from Mesolithic sites spread throughout a wide area of southern England (Jacobi *et al.* 1980; the types have been grouped and only some of the sites mentioned in this paper are used here, for reasons of exposition).

Both the matrices are examples of *Q-Matrices* in the sense that as one moves down each column the quantities never strictly decrease and then strictly increase again. The order of the rows in such a matrix gives rise to an order of the corresponding provenances which is a candidate for their relative chronological ordering (see for example, Robinson 1951, Brainard 1951, Kendall 1969). In the first case this is confirmed by the stratigraphic order in which they were found and, indeed, this illustrates the general theory well. In the second there is no such stratigraphy, far from it; and it is open to the archaeologist to accept the suggested chronology or not. Notice that if the rows of a Q-matrix are completely inverted then the matrix is still a Q-matrix. It is up to the archaeologist to decide between the two orders, or to reject both.

Both the matrices are in Q-form and the theory of seriation would tend to leave it there. But perhaps we should look for something more in the data corresponding to the fact that one is derived from a well defined stratigraphic sequence whilst the other is from a series of sites spread throughout a large area. Perhaps one should be a 'tightly' seriated matrix and the other a 'loosely' seriated one—whatever these terms may mean! I want to show in this paper that there is a sense in which this is so. I add that this idea did come into being in this way; it came from purely theoretic considerations, but it does seem to me that the results do have this interpretation (Laxton 1987, to appear).

5.2 The mathematical problem

Recall that an $m \times n$ abundance matrix $A = (r_1, r_2, \ldots, r_m)$, where the r_i are the rows of A, is said to be a *pre-Q-matrix* if there is a permutation of its rows such that the resulting $m \times n$ matrix $B = (r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(m)})$, is a Q-matrix. B is called a *Q-form* of A or resulting from A. Thus the general problem in seriation is to determine if an abundance matrix is a pre-Q-matrix or not and, if it is, to find a permutation π of its rows to cast it into a Q-form. If A is pre-Q

Rows	Proportions					Levels (top to bottom)
r_1	4	6	86	0	4	*d*
r_2	6	14	76	0	4	*e*
r_3	8	19	70	1	2	*f*
r_4	18	49	30	3	0	*g*
r_5	23	54	20	3	0	*h*
r_6	32	49	5	14	0	*i*
r_7	39	43	0	18	0	*j*
r_8	49	30	0	21	0	*k*
r_9	62	20	0	18	0	*l*
r_{10}	87	3	0	10	0	*m*

Table 5.1: Proportions of five different types of painted pottery contained in levels in the western Mound at Awatovi (Burgh 1959)

Rows	Proportions						
b_1	85	3	12	0	0	0	0
b_2	67	10	23	0	0	0	0
b_3	26	40	8	0	0	26	0
b_4	26	29	8	3	0	33	1
b_5	20	3	4	42	18	0	13
b_6	20	1	4	13	58	0	4

Table 5.2: Proportions of seven different groups of microliths from Mesolithic sites in southern England (Jacobi *et al.* 1980)

and there is only one associated Q-form B, apart from complete invertion of its rows, then B is called a *unique Q-matrix* or *Q-form.* Thus consider the following formal examples:

$$\begin{pmatrix} 0 & 90 & 10 \\ 10 & 0 & 90 \\ 90 & 10 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 90 & 10 \\ 10 & 10 & 80 \\ 100 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 90 & 10 \\ 10 & 10 & 80 \\ 80 & 10 & 10 \end{pmatrix}.$$

The first is not pre-Q; the second is a unique Q-matrix; and the third is a non-unique Q-matrix since the second and third rows can be interchanged leaving the matrix in Q-form.

Clearly if a matrix A is a Q-matrix then any subset of k of its rows form a $k \times n$ Q-matrix. In particular, any subset of three of its rows and any subset of four of its rows are Q-matrices. Therefore a *necessary* condition that a matrix be pre-Q is that every set of three of its rows and every set of four of its rows is pre-Q. Now, this condition is *not* sufficient. For example, in the matrix

$$\begin{pmatrix} 1&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&1 \end{pmatrix}$$

every subset of three and of four of its rows is pre-Q but the whole matrix is not.

However, in 1976 I proved the following theorem (Laxton 1976).

Theorem 1 *Suppose that A is an $m \times n$ abundance matrix which satisfies the following conditions:*

i) Each subset of three rows is pre-Q and its resulting Q-form is a unique $3 \times n$ Q-matrix.

ii) Each subset of four rows is pre-Q.

Then A is pre-Q.

In fact the result is actually stronger than this.

Corollary 1 *Under the conditions of Theorem 1, A has a unique Q-form.*

In itself this result is of some interest in that it reduces a search for pre-Q-ness in a $m \times n$ matrix from one which is exponential in m to one which is only polynomial in m—$O(m^4)$. The proof of this theorem is rather a long and rather tedious one by induction on the number of rows and treating many special cases. The idea for such a result, though not the proof, comes from the classical theorems of Menger (Menger 1928, Blumenthal 1953). Actually the same result had been proved by Fishburn using the language of ternary relations on sets (Fishburn 1971, 1985). The two results are entirely equivalent and the proofs very similar.

Both these results suffer from two important deficiences.

1. The entire condition (i) of the theorem is not necessary for the conclusion. For example, the matrix

$$A = \begin{pmatrix} 1&0&0 \\ 1&1&0 \\ 1&1&1 \\ 0&1&1 \end{pmatrix}$$

is a unique Q matrix but the first three rows have two distinct Q-forms

$$\begin{pmatrix} 1&0&0 \\ 1&1&0 \\ 1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&0 \\ 1&1&1 \\ 1&0&0 \end{pmatrix}.$$

On the other hand, without something to replace it the result does not stand, as our previous example showed.

2. The uniqueness of the Q-form which results under these conditions is not always desirable in practice. A set of data may give rise to several Q-forms—apart from inversion, that is.

These two problems led me to search for a more general and complete result and this is outlined in the next section.

5.3 The general theorem: fixing numbers and linear rigidity

Menger proved that if every four elements of a semi-metric space A of m points can be embedded on the Euclidean line, then all m points can. His proofs depend very much on the properties of Euclidean space, which are not available to us here in this problem. Actually Menger proved something a bit stronger. He showed that if there exist two points P and Q in A such every set of four points containing P and Q can be embedded on the Euclidean line, then all the points of A can be embedded on the line. Taking this hint as a possible generalization I have been able to prove the following result.

Theorem 2 *Let A be an $m \times n$ abundance matrix with rows $a, b, ..., m$ ($m \geq 3$).*
Let $u_1, u_2, ..., u_k$ be $k \geq 2$ distinct rows of A, where if $k > 2$, then the $k \times n$ matrix $(u_1, u_2, , ..., u_k)$ is a Q-matrix.
[Comment. In what follows in the statement of this theorem these u_i will always appear in this *fixed* order. By $(u_1, ..., u_i, ..., u_k)$ is meant the $(k-1) \times n$ matrix with the rows in this order but with precisely the row u_i missing.]
Now assume that the following conditions hold for all i and all rows $a, b, c \in A \setminus \{u_1, ..., u_k\}$:

i) the $(k+1) \times n$ matrices $(a, u_1, ..., u_k)$ and $(a, b, u_1, ..., u_i, ..., u_k)$ are pre-Q and have unique Q-forms;

ii) the $(k+2) \times n$ matrices $(a, b, c, u_1, ..., u_k)$ and $(a, b, u_1, ..., u_i, ..., u_k)$ are pre-Q.

Then the matrix $A = (a, b, ..., m)$ is a pre-Q matrix and it is has exactly one associated Q-form with the $u_1, ..., u_k$ appearing in this order.

5.4 Comments

1. The best way to state and prove this result is in terms of ternary relations. Furthermore, in this more general form the proof becomes more transparent since one is able to use the ternary relation on the rows and the u_i to define a binary relation on the rows and then one is able to use this well-established theory to complete the proof.

2. The result is both necessary and sufficient since if A is itself pre-Q, then we can take $k = m$ and the $u_1, ..., u_m$ to be all the rows of A in an order of a Q-form of A. The conditions then become vacuously true.

3. Theorem 1 is a special case of Theorem 2 when $k = 2$ and for *any* two rows u_1 and u_2 of A the conditions are valid.

4. Most importantly, the uniqueness of the resulting Q-form of A is only relative to the order of the $u_1, ..., u_k$. If the conditions held for another order of these then another Q-form would result. Again if another set of rows could be used then another Q-form might result.

Definition. A set of rows $u_1, ..., u_k$ satisfying the conditions of the theorem is called a *fixing k-string* or a *fixing string* for A. The least k for which A has a fixing k-string is called the *fixing number* of A and denoted $p = p(A)$. If A has m rows then $r = r(A) = m - p$ is called the *linear rigidity* of A. Notice that the least possible value for the fixing number is 2 and that consequently the greatest possible value for the linear rigidity is $m - 2$.

Consider the following formal examples:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

with, respectively, parameters

$$p = 5, r = 0; p = 4, r = 1; p = 3, r = 2; p = 2, r = 3.$$

5.5 Archaeological examples

In the case of the 10×5 abundance matrix recording five types of painted pottery from the undisturbed stratigraphically ordered levels at Awatovi, one can show that the pair of rows corresponding to the levels g,m is a fixing pair for the whole Q-matrix. This is the smallest it could possibly be and therefore the rigidity is $10 - 2 = 8$ *i.e.* the most it could possibly be. It is in this sense that I want to say that this data matrix is very 'strongly' seriated. Another way of looking at this is that if initially the archaeologist had from some external evidence argued that site m was definitely earlier than site g (it is obvious here but imagine the data divorced from the stratigraphy) then each other site is uniquely seriated with respect to these two! 'The relative clock is defined by these two.'

In the case of the 6×7 abundance matrix recording proportions of different types of microliths from Mesolithic sites spread over a wide area of southern England, one can show that there are no fixing pairs and no fixing triples of rows. The four rows corresponding to the sites b_1, b_2, b_4 and b_5 is a defining quadruple of rows (not all quadruples do so). Thus the fixing number is 4 and the rigidity is $6 - 4 = 2$. This is a low value and again it is in this sense that I want to say that this data matrix is very 'loosely' seriated. It takes four of the six rows uniquely to seriate the other two! At least in this case we have some external evidence to quote, for the ^{14}C dates of the sites b_1, b_2 and b_5 are 7500bc, 7000bc and 6400bc, respectively. It would be nice to have the fourth date but probably more importantly my result points to getting data from more sites to try to decrease the fixing number in an enlarged set (more data are available, see the article by Jacobi *et al.* 1980).

Perhaps the approach outlined here will encourage us to accept external evidence for the chronological ordering of sites and to incorporate it into the seriation process in some minimal way such as suggested here.

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